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Love waves in electrostrictive dielectric media

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SUMMARY
The possibility of the propagation of Love waves in an electrostrictive dielectric medium is investigated. It is shown that such waves can propagate, but the electric surface potential introduces some other features.

1. Introduction

There has been much experimental investigation [1, 2, 3] on the effect of electrostriction on dielectric solids. But, the rigorous mathematical formulation of the theory and the solution of particular problems based on such theory seems to be very rare. Recently Knops [4] discussed reciprocal theorems of electrostriction by extending Betti's reciprocal theorem of the classical elasticity. Paria [5] investigated the problem of bending of a clamped plate in the light of plane strain. He [6] also solved the problem of the propagation of disturbances in a semi-infinite electrostrictive dielectric medium.

In the present paper, the possibility of the propagation of Love waves is investigated. It is shown that such waves can propagate, but the electric potential introduces some other features (as stated later in the conclusion).

Incidentally, it may be of interest to indicate the technical or physical significances of a dielectric electrostrictive half space covered by a similar layer. For instance, the laminated sheets of electrostrictive dielectric materials are used as coatings of conductors and some components of instruments. If two sheets are used such that the thickness of one is much larger compared to the thickness of the other, then, we get a situation where the thicker sheet may be considered to be a half space and the thinner sheet may be taken as a layer over it. It has been shown in this paper that such types of combination of electrostrictive dielectric sheets can propagate Love waves also.

2. General theory

We consider an electrostrictive dielectric solid whose elastic and electric properties are homogeneous and isotropic when there is no stress or no strain. When the deformation is produced, the electric properties may be come anisotropic, and in such situations the coefficients of anisotropy depend upon the strain developed within the medium.

In such media, the stress tensor σ_{ij} is related to the strain tensor e_{ij} and the electric field E_i as [4]

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + aE_i E_j + bE_m E_m \delta_{ij}\,, \tag{2.1}$$

where the symbols have their usual meanings. By definition we have

$$e_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j})\,. \tag{2.2}$$

In the anisotropic medium, the electric displacement D_i is related to E_i as

$$D_i = k_{ij}E_j \tag{2.3}$$

where the anisotropic coefficients k_{ij} in the isothermal conditions are given by

$$k_{ij} = k\delta_{ij} + K_1 e_{ij} + K_2 e_{kk}\delta_{ij}\,. \tag{2.4}$$

The constants K, K_1 and K_2 are characteristic of the electrostrictive property and are determined experimentally. It may be shown that [3] the constants a and b in (2.1) are expressible in terms of these quantities as

$$a = 2K - K_1\,, \quad b = -(K_1 + K_2)\,. \tag{2.5}$$

The stress equations of motion are

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho X_i = \rho \frac{\partial^2 u_i}{\partial t^2}\,, \tag{2.6}$$

and Maxwell's electrical equations are

$$\operatorname{curl} \boldsymbol{E} = 0\,, \tag{2.7}$$

$$\operatorname{div} \boldsymbol{D} = 0\,. \tag{2.8}$$

Equations from (2.1) to (2.8) are the fundamental equations. They are to be solved under prescribed electrical and mechanical boundary and initial conditions.

3. Love waves

We consider a semi-infinite medium bounded by the plane $z=0$, and the positive direction of the z-axis is taken into the medium. A layer of thickness T of different material is placed over the surface $z=0$ so that the upper surface of the layer is given by $z=-T$. We consider the possibility of the propagation of a purely transverse wave of Love type in the medium such that the disturbance penetrates only a little distance into the interior. Let the wave be propagated parallel to the x-axis with velocity c. We assume that the displacement components are

$$u_1 = 0\,, \quad u_3 = 0\,, \quad u_2 = F(z)\exp\{ip(x-ct)\}\,, \tag{3.1}$$

where F is a function of z only and p is a constant, and where $i=\sqrt{-1}$. From (2.2) we get

$$\begin{aligned} e_{11} &= e_{22} = e_{33} = 0\,. \quad e_{31} = 0\,, \\ e_{12} &= \frac{ip}{2} F(z)\exp\{ip(x-ct)\}\,, \\ e_{23} &= \tfrac{1}{2}\frac{dF}{dz}\exp\{ip(x-ct)\}\,. \end{aligned} \tag{3.2}$$

The solution of (2.7) is

$$\boldsymbol{E} = -\operatorname{grad}\phi\,.$$

We assume that

$$\phi = V(z)\exp\left\{\frac{ip}{2}(x-ct)\right\}$$

where V is a function of z. Hence

$$\left.\begin{aligned} E_1 &= -\frac{ip}{2} V \exp\left\{\frac{ip}{2}(x-ct)\right\}, \quad E_2 = 0\,, \\ E_3 &= -\frac{dV}{dz}\exp\left\{\frac{ip}{2}(x-ct)\right\}. \end{aligned}\right. \tag{3.3}$$

From (2.4) we get

$$k_{11} = K\,, \quad k_{22} = K\,, \quad k_{33} = K\,, \quad k_{31} = 0\,,$$

$$k_{12} = \frac{iK_1 p}{2} F(z) \exp\{ip(x-ct)\}\,,$$

$$k_{23} = \tfrac{1}{2} K_1 \frac{dF}{dz} \exp\{ip(x-ct)\}\,. \tag{3.4}$$

Using (3.3) and (3.4) in (2.3) we get

$$D_1 = -\frac{ipk}{2} V \exp\left\{\frac{ip}{2}(x-ct)\right\},$$

$$D_2 = \left[\tfrac{1}{4} k_1 p^2 V(z) F(z) - \tfrac{1}{2} k_1 \frac{dV}{dz}\frac{dF}{dz}\right] \exp\{\tfrac{3}{2} ip(x-ct)\} \tag{3.5}$$

$$D_3 = -k \frac{dV}{dz} \exp\left\{\frac{ip}{2}(x-ct)\right\}.$$

Hence (2.8) gives

$$\frac{d^2 V}{dz^2} - \frac{p^2}{4} V = 0\,.$$

Its solution is

$$V = V_0 \exp\left\{-\frac{p}{2}(z+T)\right\}, \qquad (-T \leqq z \leqq \infty)\,, \tag{3.6}$$

where V_0 is the value of V at the surface $z=-T$. Now using (3.2), (3.3) and (3.6) in (2.1), we get

$$\sigma_{11} = (a+2b)\frac{p^2}{4} V_0^2 \exp\{-p(z+T)\} \exp\{ip(x-ct)\}\,,$$

$$\sigma_{22} = b\frac{p^2}{4} V_0^2 \exp\{-p(z+T)\} \exp\{ip(x-ct)\}\,,$$

$$\sigma_{33} = (a+2b)\frac{p^2}{4} V_0^2 \exp\{-p(z+T)\} \exp\{ip(x-ct)\}\,,$$

$$\sigma_{13} = -\frac{iap^2}{4} V_0^2 \exp\{-p(z+T)\} \exp\{ip(x-ct)\}\,. \tag{3.7}$$

We also have

$$\sigma_{12} = i\mu p F(z) \exp\{ip(x-ct)\}\,,$$

$$\sigma_{23} = \mu \frac{dF}{dz} \exp\{ip(x-ct)\}\,. \tag{3.8}$$

From (3.7), it follows that the components of the normal stress and the shearing stress σ_{13} depends upon the electric potential V_0 and are independent of the mechanical deformations.

Now, two of the equations (2.6) are satisfied if the body forces X_1 and X_3 have values

$$X_1 = -\frac{1}{2\rho}(a+b)p^3 V_0^2\, i \exp\{-p(z+T)\} \exp\{ip(x-ct)\}\,,$$

$$X_3 = \frac{1}{2\rho} bp^3 V_0^2 \exp\{-p(z+T)\} \exp\{ip(x-ct)\}\,, \tag{3.9}$$

while the remaining one is satisfied in the absence of X_2 if

$$\frac{d^2F}{dz^2} + p^2\left(\frac{c^2}{\beta^2} - 1\right) z = 0\,, \tag{3.10}$$

where

$$\beta^2 = \mu/\rho\,. \tag{3.11}$$

The solution of (3.10) can be written as

$$F = A\cos(sz) + B\sin(sz)\,, \qquad (-T \leqq z \leqq 0)\,, \tag{3.12}$$

$$F = A\exp(-s'z)\,, \qquad (0 \leqq z \leqq \infty)\,, \tag{3.13}$$

where

$$s = p\left(\frac{c^2}{\beta^2} - 1\right)^{\frac{1}{2}}, \quad s' = p\left(1 - \frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}}, \quad \beta_1^2 \equiv \frac{\mu_1}{\rho_1}. \tag{3.14}$$

Here (μ_1, ρ_1) are the values of (μ, ρ) for the material in the range $(0 < z < \infty)$, and s' is real and positive so that $c < \beta_1$.

Now, the conditions of continuity of the stresses σ_{33} and σ_{13} at the interface $z = 0$ imply that the value of a and b in the range $-T < z < 0$ must be the same as those for the range $0 < z < \infty$. The continuity of σ_{23} at $z = 0$ implies

$$sB\mu = -s'A\mu'\,. \tag{3.15}$$

From (3.7), it follows that the stresses σ_{33} and σ_{13} have non-zero values on the surface $z = -T$ introduced by the potential V_0. The stress σ_{23} at $z = -T$ is zero if

$$A\sin(sT) + B\cos(sT) = 0\,. \tag{3.16}$$

From (3.15) and (3.16), we get the wave velocity equation

$$s\mu\tan(sT) = s'\mu'\,, \tag{3.17}$$

for a prescribed wave number p. Since the right-hand side of (3.17) is real and positive, the left-hand side must be also real and positive, which implies that s is real and positive, i.e., $\beta < c$. Thus we have $\beta < c < \beta_1$, which is the classical result.

4. Conclusions

It is shown that Love waves can propagate in an electrostrictive dielectric material if the modulus of rigidity and the density of the layer are different from those of the underlying material, but the electric properties of both material must be the same. The prescribed electric potential at the upper surface introduces a body force. The component of this body force in the direction of the mechanical displacement is however zero. The potential also introduces the normal stresses and one component of the shear stresses. The normal stress and one component of shear on the upper surface have non-zero values and are to be balanced with corresponding prescribed surface forces. The wave numbers of the mechanical displacement $\boldsymbol{u}$ and stresses σ_{ij} have the same value p, whereas the electric potential ϕ and electric fields $\boldsymbol{E}$ have wave numbers equal to $\frac{1}{2}p$. The electric displacements D_1 and D_3 have the wave numbers $\frac{1}{2}p$ whereas D_2 has the wave number $\frac{3}{2}p$. Thus Love wave can propagate along with these above features.

If the layer thickness $T \to 0$, then equation (3.17) shows that $s' = 0$ $(\mu' \neq 0)$. Equation (3.16) shows that $B = 0$. Equation (3.15) is then identically satisfied. Equation (3.13) shows that $F(z) = A$, a constant. Equation (3.12) then shows that $A = 0$. Hence the equation (3.1) leads to $u_2 = 0$. Thus, there cannot be any Love wave in an electrostrictive half space without a superimposed layer.

Lastly, a brief remark on the mathematical technique used in the solution of the problem is also interesting. Relation (2.1) as well as the relation (2.3) together with (2.4) show that the phenomenon considered in this paper is non-linear. But the technique adopted in section 3 to obtain the solution is the one usually used in the corresponding elastic problem which is linear. Thus, we get here an illustration in which the linear technique is successful in the non-linear phenomenon also.

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